

Cospectral lifts of graphs

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Abstract

We prove that for a pair of cospectral graphs G and H , there exist their non trivial lifts G' and H' which are cospectral. More over for a pair of cospectral graphs on 6 vertices, we find some cospectral lifts of them.

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1 Introduction

Let $G = (V, E)$ be a simple graph on the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set E . The *adjacency matrix* of G is an n by n matrix $A(G)$ whose (i, j) -th entry is 1 if vertices v_i and v_j are adjacent and 0, otherwise. The *spectrum* of G is the multi-set of eigenvalues of $A(G)$. Two graphs G and G' are called *cospectral* if they share the same spectrum. We say G is *determined by spectrum* (*DS* for short) if it has no non-isomorphic cospectral mate.

The problem of constructing cospectral graphs, has been investigated by some authors. For a survey of results on this area we refer the reader to [2, 3, 6]. In [4] the authors have used the concept m -cospectrality to construct new cospectral graphs. Haemers et al in [1] have considered Godsil-McKay switching method to construct non-isomorphic cospectral graphs see the paper for more details. In this article we use the concept lift of graphs to construct new non-isomorphic cospectral graphs from given small cospectral pairs of graphs.

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2 Preliminaries

In this section we mention some basic definitions and results which will be used during the paper. We denote by \dot{E} the set of all ordered pairs $\{(i, j) \mid i < j, \{v_i, v_j\} \in E\}$. For an Abelian group of order k , say Gr , a k -Abelian signature s of the graph G is a map $s : \dot{E} \rightarrow Gr$. A k -Abelian lift of the graph G , associated with the signature s , which is denoted by $G(s)$, is a graph on the vertex set $V(G) \times [k]$ ($[k] = \{0, 1, \dots, k-1\}$, $Gr = (\{g_0, g_1, \dots, g_{k-1}\}, *)$), where for any $(i, j) \in \dot{E}$ and $a, b \in [k]$ there is an edge between (v_i, a) and (v_j, b) if and only if $s(i, j) * g_a = g_b$. Note that in the graph $G(s)$, for any $(i, j) \in \dot{E}$, there is a matching between the vertex sets $V_i = \{v_i\} \times [k]$ and $V_j = \{v_j\} \times [k]$. If a graph have m edges there may be k^m different k -Abelian lifts of G , since the sets V_i, V_j are matched in k different ways. If the signature s maps all pairs to the same element $g \in Gr$, then we denote the corresponding lift $G(s)$ with G_g . We illustrate the definition of the k -lifts of a graph in the following figure. In the following graph the graph G is the cycle C_4 , and the corresponding signature is $s : \dot{E} \rightarrow \mathbb{Z}_2$, with $s(1, 3) = 0, s(1, 4) = 0, s(2, 3) = 1, s(2, 4) = 0$.

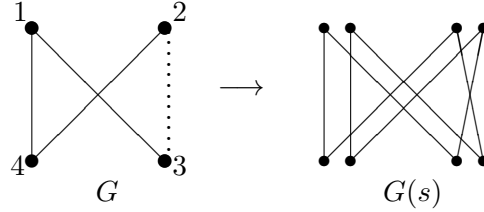


Figure1. 2-lift of G corresponding to the signature s

Let $Gr = (\{g_1 = 1, g_2, \dots, g_n\}, *)$ be a group of order n . For any group element say $g \in Gr$ there is an $n \times n$ permutation matrix P_g in correspondence, which is defined bellow,

$$P_g(i, j) = \begin{cases} 1 & \text{if } g_i * g = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 The function $\phi : Gr \rightarrow SL(n, \mathbb{R})$, where $SL(n, \mathbb{R})$ is the set of $n \times n$ real non-singular matrices and $\phi(g) = P_g$, is a group homomorphism.

The eigenvalues of the graph $G(s)$ has been studied in the literature. For instance in the following theorem from [5] the authors have obtained the eigenvalues of Abelian t -lifts. See [5] for more details and the notations.

Theorem 1 Let G be a multigraph and ϕ be a signature assignment to an Abelian group. Let β be a common basis of eigenvectors of the permutation matrices in the image of ϕ . For

every $\mathbf{x} \in \beta$, let $A_{\mathbf{x}}$ be the matrix obtained from the adjacency matrix of G by replacing any (u, v) -entry of $A(G)$ by $\sum_{(e, u, v) \in \vec{E}(G)} \lambda_{\mathbf{x}}(\phi(e, u, v))$. Then the spectrum of the t -lift $G(\phi)$ of G is the multiset union of the spectra of the matrices $A_{\mathbf{x}}(\mathbf{x} \in \beta)$.

3 Main result

Our main problem here is "for given pair of cospectral graphs G and H , is there k -Abelian signatures s, s' which $G(s)$ and $H(s')$ are cospectral?". We look for general answers of this question.

It is known that for $l, l' \in Gr$ the permutation matrices $P_l, P_{l'}$ commute, so they have common basis of eigenvectors. The following theorem is a straight consequence of Theorem 1.

Theorem 2 Let G be a graph and s be a k -cyclic signature of G . Let β be a common basis of eigenvectors of the permutation matrices in the image of s . For every $\mathbf{x} \in \beta$, let $A_{\mathbf{x}}$ be the matrix defined bellow

$$A_{\mathbf{x}}(i, j) = \begin{cases} \lambda_{\mathbf{x}}(P_{s(i, j)}) & i < j, \\ 0 & i = j, \\ \lambda_{\mathbf{x}}^{-1}(P_{s(i, j)}) & i > j. \end{cases}$$

Then the spectrum of $G(s)$ is the multi-set union of the spectra of the matrices $A_{\mathbf{x}}(\mathbf{x} \in \beta)$.

Lemma 2 Let Gr be a group of order n . For any $g \in Gr$, any eigenvalues of the permutation matrix P_g is an n 'th root of unity.

Proof. The assertion follows by the fact that the order of any element in the group divides the order of group. Hence $g^n = 1$, and therefore $P_g^n = I_n$. Hence the minimal polynomial of P_g , say $m(P_g, x)$ divides the polynomial $x^n - 1$, thus the assertion follows. \square

Lemma 3 If G and H are cospectral graphs on n verices and Gr be a finite group of order t . If for $g \in Gr$ the matrix P_g is symmetric, then the graphs G_g and H_g are cospectral.

Proof. Since the signature corresponds the fixed element g to all the edges of the graph G , and $P_g^{-1} = P_g$, then by Theorem 1, the eigenvalues of the graph G_g are the multi-set union of the matrices $\omega_i A(G)$, where ω_i 's are the eigenvalues of P_g for $i = 1, 2, \dots, n$. On the other hand the eigenvalues of $\omega_i A(G)$ are $\omega_i \lambda_j$ where λ_j is the j 'th eigenvalue of G . Hence the spectrum of G_g and H_g are the multi-set $\{\omega_i \lambda_j\}_{i=1, \dots, n}^{j=1, \dots, n}$. \square

3.1 Examples

We consider two cospectral graphs G and H , shown in the Figure 1. We try to find possible Abelian lifts of them say $G(s)$ and $H(s')$ which are also cospectral.

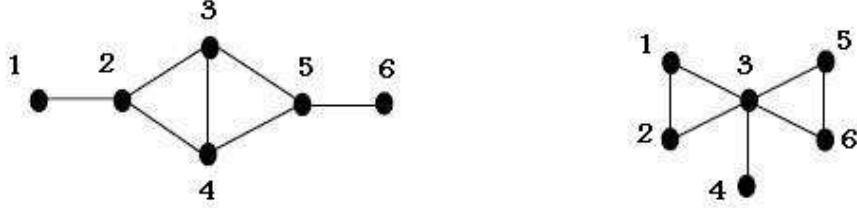


Figure1. Two cospectral graphs G and H

We first consider all possible t -Abelian signatures of the graphs G and H , suppose that the matrices $A(G)_x, A(H)_x$, corresponding to the graphs G, H and their prescribed signatures, which is introduced in Theorem 1 are of the following general forms,

$$A(G)_x = \begin{pmatrix} 0 & u & 0 & 0 & 0 & 0 \\ u^{-1} & 0 & v & w & 0 & 0 \\ 0 & v^{-1} & 0 & x & y & 0 \\ 0 & w^{-1} & x^{-1} & 0 & z & 0 \\ 0 & 0 & y^{-1} & z^{-1} & 0 & r \\ 0 & 0 & 0 & 0 & r^{-1} & 0 \end{pmatrix}, A(H)_x = \begin{pmatrix} 0 & u_1 & v_1 & 0 & 0 & 0 \\ u_1^{-1} & 0 & w_1 & 0 & 0 & 0 \\ v_1^{-1} & w_1^{-1} & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & x_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & y_1^{-1} & 0 & 0 & r_1 \\ 0 & 0 & z_1^{-1} & 0 & r_1^{-1} & 0 \end{pmatrix}.$$

note that $r, u, v, \dots, z, r_1, u_1, v_1, \dots, z_1$ are the complex variables and stand for the eigenvalues of the permutation matrices corresponding to each edge. Using Theorem 2 we find sufficient conditions on the signatures such that the corresponding lifts become cospectral.

Theorem 3 *Let s, s' be k -Abelian lifts on the graphs G, H respectively. If the following situations hold, the graphs $G(s)$ and $H(s')$ are cospectral.*

- $\frac{w}{v} = \frac{y}{z}$
- $2\left(\frac{xv}{w} + \frac{w}{xv}\right) = \frac{y_1 r_1}{z_1} + \frac{z_1}{y_1 r_1} + \frac{u_1 w_1}{v_1} + \frac{v_1}{u_1 w_1}.$

Proof. We consider all possibilities for the matrices $A(G)_x, A(H)_x$ corresponding to the graphs G, H . Comparing the coefficients of $\chi(A(G)_x, t), \chi(A(H)_x, t)$, the equality holds if and only if

$$2 = \frac{wz}{vy} + \frac{vy}{wz}, \quad (1)$$

(2)

cospectral lifts of the graphs G and H .

$$z = \frac{vy}{w}, \quad u_1 = y_1 = x, \quad w_1 = r_1 = v, \quad z_1 = w$$

the cyclic representation.

$$\begin{aligned}s(1, 2) &= (1, 2, 3), s(2, 3) = (1, 3, 2), s(2, 4) = \text{id}, \\s(3, 4) &= (1, 3, 2), s(3, 5) = (1, 3, 2), s(4, 5) = (1, 2, 3), s(5, 6) = (1, 2), \\s'(1, 2) &= (1, 3, 2), s'(1, 3) = \text{id}, s'(2, 3) = (1, 3, 2), \\s'(3, 4) &= \text{id}, s'(3, 5) = (1, 3, 2), s'(3, 6) = \text{id}, s'(5, 6) = (1, 3, 2)\end{aligned}$$

The adjacency matrices of the graphs $G(s)$ and $H(s')$ are of the following forms.

$$A(G(s)) = \begin{pmatrix} 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0 \\ 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0 \\ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0 \\ 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1 \\ 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0 \\ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 \end{pmatrix}, A(H(s')) = \begin{pmatrix} 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 \\ 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1 \\ 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0 \\ 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 \\ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0 \end{pmatrix}.$$

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